12. 02. 2024 {Some Impt. Sums}

Q. If $r = |\vec{r}|$ and $\vec{r} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ then prove that $\nabla \log|\vec{r}| = \frac{\vec{r}}{r^2}$.

Proof

$$LHS = \nabla \log|\vec{r}| = \nabla \log r$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\log r$$

$$= \vec{i}\frac{\partial}{\partial x}(\log r) + \vec{j}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)$$

$$= \vec{i}\frac{1}{r}\frac{\partial r}{\partial x} + \vec{j}\frac{1}{r}\frac{\partial r}{\partial y} + \vec{k}\frac{1}{r}\frac{\partial r}{\partial z}$$

$$= \frac{1}{r}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right) \quad \text{--- (1)}$$

Now $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ (Given)

$$\Rightarrow r^2 = x^2 + y^2 + z^2 \quad \text{--- (2)}$$

Differentiating eq(2) partially w.r.to x, keeping y and z constant, we have

$$2r\frac{\partial r}{\partial x} = 2x \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$ --- (3)

using (3) in (1), we have

$$LHS = \frac{1}{r}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= \frac{1}{r^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{\vec{r}}{r^2}$$

$[\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}]$

= RHS Proved

Date
Page

2.(a) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then
$$\operatorname{div}(r^n \vec{r}) = (n+3) r^n.$$

Proof

$$LHS = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(r^n\vec{r})$$

$$= r^n\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{r}$$
$$+ \left[\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)r^n\right]\cdot\vec{r}$$

$$= r^n\left[\vec{i}\cdot\frac{\partial\vec{r}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{r}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{r}}{\partial z}\right]$$
$$+ \left(\vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}\right)\cdot\vec{r} \quad \text{--- (1)}$$

Now $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow \frac{\partial\vec{r}}{\partial x} = \vec{i},\ \frac{\partial\vec{r}}{\partial y} = \vec{j},\ \frac{\partial\vec{r}}{\partial z} = \vec{k} \quad \text{--- (2)}$$

Also $r^2 = x^2 + y^2 + z^2$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{--- (3)}$$

Also $\frac{\partial r^n}{\partial x} = n r^{n-1}\cdot\frac{\partial r}{\partial x} = n r^{n-1}\cdot\frac{x}{r}$

$$\Rightarrow \frac{\partial r^n}{\partial x} = n x r^{n-2}. \quad [\text{using (3)}]$$

Similarly $\frac{\partial r^n}{\partial y} = n y r^{n-2}$, $\frac{\partial r^n}{\partial z} = n z r^{n-2}$ — (4)

Using (2) & (4) in eq (1), we get

$\therefore$ LHS $= r^n (\vec{i}\cdot\vec{i} + \vec{j}\cdot\vec{j} + \vec{k}\cdot\vec{k})$

$+ (n x r^{n-2}\vec{i} + n y r^{n-2}\vec{j} + n z r^{n-2}\vec{k})\cdot$

$= r^n (1+1+1) + n r^{n-2} (x\vec{i} + y\vec{j} + z\vec{k})\cdot\vec{r}$

$= 3 r^n + n r^{n-2} (\vec{r}\cdot\vec{r})$

$= 3 r^n + n r^{n-2} \cdot r^2$

$= 3 r^n + n r^n = r^n (n+3)$

$=$ RHS.

Definition Solenoidal vector

A vector $\vec{V}$ is called solenoidal if $\operatorname{div}\vec{V} = 0$ i.e. $\nabla\cdot\vec{V} = 0$.

$\boxed{2(b)}$ Prove that $r^n \vec{r}$ is solenoidal when $n = -3$.

Soln If $r^n \vec{r}$ is solenoidal

$\Rightarrow \operatorname{div} r^n \vec{r} = 0$

$\Rightarrow \nabla\cdot r^n \vec{r} = 0$ [From the above sum]

$\Rightarrow (n+3) r^n = 0$ but $r^n \neq 0$

$\Rightarrow n+3 = 0 \Rightarrow \boxed{n = -3}$